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Signatures of finite EIII, EVI, EVII, EIX representations

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Abstract. The paper deals with the real exceptional Lie algebras EIII, EVI, EVII, EIX and their arbitrary irreducible representations. Hermitian forms invariant relative to these representations are considered. The existing signature formulae for these forms are improved.

1. Introduction

Consider an irreducible representation $\varphi\mathfrak{g} \to \mathfrak{sl}(V)$ of simple complex exceptional Lie algebra $\mathfrak{g}$. If $\mathfrak{g}_\sigma$ is a real form of inner type for $\mathfrak{g}$, then $\varphi(\mathfrak{g}_\sigma) \subset \mathfrak{su}(p,q)$. Let $\delta = p - q$. Signature formulae for the calculation of δ were obtained in [1]. In the case $\mathfrak{g}_\sigma = $ EII, EV and EVIII they are easy to use. As follows from this paper, it is possible to improve signature formulae in the case $\mathfrak{g}_\sigma = $ EIII, EVI, EVII, EIX. Hence the formulae for the calculation of the number of linearly independent space-like or time-like vectors in the representation space can be obtained in terms of the highest weight. The results can be used for the constructions of mathematical models in theoretical physics.

2. Definitions

Definitions used in this paper coincide with those in [1]. Let $\mathfrak{h}$ be the fixed Cartan subalgebra of the algebra $\mathfrak{g}$ and let R be the root system associated with the pair $(\mathfrak{g}, \mathfrak{h})$. Suppose $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ is a set of the simple roots of R, where $r = \mathrm{rank}(\mathfrak{g})$. Let ω_i, $i = 1, \ldots, r$ be basis representations of the algebra $\mathfrak{g}$, that is $2(\omega_i, \alpha_k)/(\alpha_k, \alpha_k) = \delta_{ik}$, where $\alpha_k \in \Pi$, $i, k = 1, \ldots, r$. Let symbol

$$\begin{array}{l} \lambda_1\ \ \lambda_3\ \ \lambda_4\ \ \lambda_5\ \qquad \lambda_r \\ 0 - 0 - 0 - 0 - \ldots - 0 \\ \qquad\quad | \\ \qquad\quad 0\ \lambda_2 \end{array}$$

denote the representation with the highest weight $\lambda = \sum_{j=1}^{r} \lambda_j \omega_j$. Let R^+ be the set of all positive roots relative to $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r$ and let $\rho = \frac{1}{2} \cdot \sum_{\beta \in R^+} \beta$ be half the sum of the positive roots R^+. Denote by W the Weyl group of R. For any $\beta \in R$ denote by s_β the reflection defined by the root β, that is $s_\beta(\boldsymbol{v}) = \boldsymbol{v} - 2[(\beta, \boldsymbol{v})/(\beta, \beta)]\beta$. Let $X(\lambda) = \{\beta | \beta \in R^+, (\beta, \frac{1}{2}(\lambda + \rho)) \in \mathbb{Z}\}$. Card $(X(\lambda))$ must be taken from table 1. Let $C_\lambda = \prod_{\beta \in X(\lambda)} (\beta, \frac{1}{2}(\lambda + \rho))$. The root realizations coincide with those in [1] and [2]. Namely, for any algebra in question the particular basis $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_8$ was given there.

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3. The case $\mathfrak{g} = \mathbf{E_6}$, $\mathfrak{g}_\sigma = \mathbf{EIII}$

By symbol $\begin{pmatrix} a & c & d & e & f \\ & & b & & \end{pmatrix}$ denote the root $\beta = a\boldsymbol{\alpha}_1 + b\boldsymbol{\alpha}_2 + c\boldsymbol{\alpha}_3 + d\boldsymbol{\alpha}_4 + e\boldsymbol{\alpha}_5 + f\boldsymbol{\alpha}_6$.

If $\mathrm{card}(X(\lambda)) = 16$, then from [1] it follows that

$$|\delta| = \frac{C_\lambda}{2^{11} \times 3^5 \times 5^2 \times 7} \frac{1}{4!} \left| \sum_{\mu_i} \cos(\pi(\mu_i - \eta_0, \lambda + \rho)) \cdot (\mu_i, \lambda + \rho)^4 \cdot \dim V^{\mu_i} \right| \tag{1}$$

where the sum in (1) embraces all weights μ_i of the representation $\begin{array}{c} 0\quad 0\quad 0\quad 0\quad 1 \\ 0 - 0 - 0 - 0 - 0 \\ | \\ 0\ 1 \end{array}$ with the highest weight $\eta_0 = \omega_2 + \omega_6$. Transform formula (1) to give

$$\lambda + \rho = \sum_{j=1}^{6} (\lambda_j + 1) \cdot \omega_j = \sum_{j=1}^{8} h_j \cdot \varepsilon_j$$

where

$$\begin{aligned}
h_1 &= \tfrac{1}{6}(3\lambda_1 + 6\lambda_2 + 6\lambda_3 + 9\lambda_4 + 6\lambda_5 + 3\lambda_6 + 33) \\
h_2 &= \tfrac{1}{6}(5\lambda_1 + 4\lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6 + 15) \\
h_3 &= \tfrac{1}{6}(-\lambda_1 + 4\lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6 + 9) \\
h_4 &= \tfrac{1}{6}(-\lambda_1 - 2\lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6 + 3) \\
h_5 &= \tfrac{1}{6}(-\lambda_1 - 2\lambda_3 - 3\lambda_4 + 2\lambda_5 + \lambda_6 - 3) \\
h_6 &= \tfrac{1}{6}(-\lambda_1 - 2\lambda_3 - 3\lambda_4 - 4\lambda_5 + \lambda_6 - 9) \\
h_7 &= \tfrac{1}{6}(-\lambda_1 - 2\lambda_3 - 3\lambda_4 - 4\lambda_5 - 5\lambda_6 - 15) \\
h_8 &= \tfrac{1}{6}(-3\lambda_1 - 6\lambda_2 - 6\lambda_3 - 9\lambda_4 - 6\lambda_5 - 3\lambda_6 - 33).
\end{aligned} \tag{2}$$

Then $\mu_i - \eta_0 = \sum_{j=1}^{6} n_{ij}\alpha_j$, where $n_{ij} \in \mathbb{Z}$ and $\alpha_j \in \Pi$, $j = 1, \ldots, 6$. Hence

$$\cos(\pi(\mu_i - \eta_0, \lambda + \rho)) = \cos\left(\pi \sum_{j=1}^{6} n_{ij}(\lambda_j + 1)\right). \tag{3}$$

Consider, in accordance with the notations of table 1, the representation $\bar{\lambda}$ of type $\begin{array}{c} \mathrm{o}\quad \mathrm{o}\quad \mathrm{o}\quad \mathrm{o}\quad \mathrm{o} \\ \mathrm{e} \end{array}$. Then $\bar{\lambda} + \rho \simeq \frac{1}{6}(60, 30, 18, 6, -6, -18, -30, -60) = (10, 5, 3, 1, -1, -3, -5, -10) \simeq (0, 1, -1, 1, -1, 1, -1, 0) \simeq (0, 1, 1, 1, 1, 1, -1, 0) \simeq (-1, 0, 0, 0, 0, 0, -2, -1) \simeq (2, 0, 0, 0, 0, 0, 0, 0)$.

The notation '$\lambda + \rho \simeq \cdots$' means here and everywhere below that the transformation does not change the value of (3). Furthermore, using the table of all weights μ_i of the representation $\begin{array}{c} 0\quad 0\quad 0\quad 0\quad 1 \\ 0 - 0 - 0 - 0 - 0 \\ | \\ 0\ 1 \end{array}$ [1], we derive from (1) the formula

$$|\delta| = \frac{C_{\bar{\lambda}}}{2^{11} \times 3^5 \times 5^2 \times 7} \left| -84h_1^4 + 12h_1^2 \cdot \sum_{j=2}^{7} h_j^2 - 4\sum_{j=2}^{7} h_j^4 + \left(\sum_{j=2}^{7} h_j^2\right)^2 \right|. \tag{4}$$

It is possible to consider in the same way the other 35 cases when $\mathrm{card}(X(\lambda)) = 16$ and to derive the table of δ similar to those in [3]. But the table would be too big in the cases $\mathfrak{g} = \mathrm{E}_7$ and $\mathfrak{g} = \mathrm{E}_8$. To overcome this we use the following lemma.

Table 1. Card$(X(\lambda))$, $\mathfrak{g} = E_6, E_7, E_8$. Symbol e (o) in the column λ_i denotes an even (odd) λ_i. Symbol a denotes any λ_i independent of whether it is even or odd.

(Representations are written as $\lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6$ / λ_2.)

	Representation $\lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6$ / λ_2		Card$(X(\lambda))$ $\mathfrak{g} = E_6$	λ_7	Card$(X(\lambda))$ $\mathfrak{g} = E_7$	λ_8	Card$(X(\lambda))$ $\mathfrak{g} = E_8$
S_{11}	a e a e a / e	e e e o a / o		e	28	a	56
	e o a e a / e	o e o o a / o	16			e	56
	a o e o a / o			o	31	o	64
S_{12}	a e a o e / e	e e e e e / o		o	28	a	56
	e o a o]e /]e	o e e e o / o					
	o o a o o / e	e e o e o / o	16				
	a o e e e / o	o e o e e / o		e	31	e	56
	a o o e o / o					o	64
S_{21}	o o a e a / e	o e e o a / o		e	36	a	64
						e	64
	e a o o a / o	o o o o e / o	20	o	31	o	56
S_{22}	a e a o o / e	e e e e o / o		o	36	a	64
	e o a o o / e	o e e e e / o					
	o o a o e / e	e e o e e / o	20				
	a o e e o / o	o e o e o / o		e	31	e	64
	a o o e e / o					o	56
S_{31}				e	36	a	64
	o o o o o / o		36			e	64
				o	63	o	120

Lemma 1. Let $\lambda = \sum_{j=1}^{6} \lambda_j \omega_j$ be the highest weight of the representation $\varphi : E_6 \to \mathfrak{sl}(V)$ and let card$(X(\lambda)) = 16$. For any such λ there exists an element w from the Weyl group W such that $w(\lambda + \rho) \simeq (2, 0, 0, 0, 0, 0, 0, 0)$. The results of the lemma are derived by straightforward calculation, using table 2. The necessary elements w are the compositions of the reflections s_β defined by the roots from the line β of table 2. For example, for λ of type $\begin{matrix} e & e & e & e & e \\ & & e & & \end{matrix}$ the root β is $\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{pmatrix}$ and $w(\lambda + \rho) = s_\beta(\lambda + \rho) \simeq (2, 0, 0, 0, 0, 0, 0, 0)$. For λ of type $\begin{matrix} e & o & o & e & e \\ & & e & & \end{matrix}$ the roots β_1 and β_2

Table 2. The element $w \in W$ such that $w(\lambda+\rho) \equiv \begin{smallmatrix}e&e&e&e&e\\&&o&&\end{smallmatrix}$ (mod 2).

The type of λ	$\begin{smallmatrix}e&e&e&e&e\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&o&e&e&e\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&e&e&o&e\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}o&e&o&e&e\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&e&o&e&o\\&&e&&\end{smallmatrix}$
The root β	$\begin{pmatrix}1&1&2&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&1&2&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&2&1&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&2&2&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&2&2&1\\&&1&&\end{pmatrix}$
$w \in W$	s_β	s_β	s_β	s_β	s_β
The type of λ	$\begin{smallmatrix}e&e&e&e&e\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}e&o&e&e&e\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}e&e&e&o&e\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}o&e&o&e&e\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}e&e&o&e&o\\&&o&&\end{smallmatrix}$
The root β	$\begin{pmatrix}0&1&1&1&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&1&1&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&1&1&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&1&1&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&1&1&0&0\\&&1&&\end{pmatrix}$
$w \in W$	s_β	s_β	s_β	s_β	s_β
The type of λ	$\begin{smallmatrix}o&o&e&o&o\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}e&o&e&o&e\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}o&e&e&e&o\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}o&o&e&o&o\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&o&e&o&e\\&&e&&\end{smallmatrix}$
The root β	$\begin{pmatrix}1&2&3&2&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&1&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&1&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&1&2&1&0\\&&1&&\end{pmatrix}$
$w \in W$	s_β	s_β	s_β	s_β	s_β
The type of λ	$\begin{smallmatrix}o&e&e&e&o\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&o&o&e&o\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}o&e&o&o&e\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&o&o&e&o\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}o&e&o&o&e\\&&o&&\end{smallmatrix}$
The root β	$\begin{pmatrix}1&2&2&2&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&1&2&2&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&2&2&1&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&1&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&1&1&1\\&&1&&\end{pmatrix}$
$w \in W$	s_β	s_β	s_β	s_β	s_β
The type of λ	$\begin{smallmatrix}o&e&e&e&e\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&e&e&e&o\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&e&o&e&e\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&o&o&e&e\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&e&o&o&e\\&&e&&\end{smallmatrix}$
The root β_1	$\begin{pmatrix}1&2&2&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&2&2&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&2&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&1&2&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&2&1&0\\&&1&&\end{pmatrix}$
The root β_2	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$
$w \in W$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$
The type of λ	$\begin{smallmatrix}o&e&o&e&o\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}o&o&e&e&e\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}e&e&e&o&o\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}o&e&e&o&e\\&&e&&\end{smallmatrix}$	$\begin{smallmatrix}e&o&e&e&o\\&&e&&\end{smallmatrix}$
The root β_1	$\begin{pmatrix}1&2&2&2&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&2&2&2&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&2&2&2&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&2&2&1&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&1&2&2&1\\&&1&&\end{pmatrix}$
The root β_2	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&1&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&1&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$
$w \in W$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$
The type of λ	$\begin{smallmatrix}e&o&e&o&o\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}o&o&e&o&e\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}o&e&o&o&o\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}o&o&o&e&o\\&&o&&\end{smallmatrix}$	$\begin{smallmatrix}e&o&o&o&e\\&&e&&\end{smallmatrix}$
The root β_1	$\begin{pmatrix}1&2&2&2&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&2&2&2&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&2&1&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&1&2&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&1&2&1&0\\&&1&&\end{pmatrix}$
The root β_2	$\begin{pmatrix}0&1&1&0&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&1&1&0\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&1&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}1&1&1&1&1\\&&1&&\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$
$w \in W$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$	$s_{\beta_2}\circ s_{\beta_1}$
The type of λ	$\begin{smallmatrix}o&o&o&o&o\\&&e&&\end{smallmatrix}$				
The root β	—				
$w \in W$	Id				

are $\begin{pmatrix}0&1&2&1&1\\&&1&&\end{pmatrix}$ and $\begin{pmatrix}0&0&0&0&0\\&&1&&\end{pmatrix}$. So $w(\lambda+\rho) = s_{\beta_2}\circ s_{\beta_1}(\lambda+\rho) =$

$s_{\beta_2}(s_{\beta_1}(\lambda+\rho)) \simeq (2,0,0,0,0,0,0,0)$. From lemma 1 it follows that it is possible to use the multinomial expression (4) for the calculation of δ. Hence from theorem 2 [1] and lemma 1 we derive the following theorem.

Theorem 1. Suppose $\mathfrak{g} = \mathrm{E}_6$, $\mathfrak{g}_\sigma = \mathrm{EIII}$ and $\lambda = \sum_{j=1}^{6} \lambda_j \omega_j$ is the highest weight of arbitrary representation $\varphi : \mathrm{E}_6 \to \mathfrak{sl}(V)$.

If $\mathrm{card}(X(\lambda)) = 20$, then $|\delta| = \frac{C_\lambda}{2^5 \times 3^5 \times 5^2 \times 7}$.

If $\mathrm{card}(X(\lambda)) = 36$, then $\delta = 0$.

If $\mathrm{card}(X(\lambda)) = 16$, then

$$|\delta| = \frac{C_\lambda}{2^{11} \times 3^5 \times 5^2 \times 7} \left| -84h_1^4 + 12h_1^2 \sum_{j=2}^{7} h_j^2 - 4\sum_{j=2}^{7} h_j^4 + \left(\sum_{j=2}^{7} h_j^2\right)^2 \right| \tag{5}$$

where $h_j,\ j = 1,\ldots,8$ are the components of the vector $w(\lambda+\rho)$ in the basis $\varepsilon_1,\ldots,\varepsilon_8$ and $w \in W$ must be taken from table 2.

4. Remarks

The elements $w \in W$ in lemma 1 and theorem 1 are not unique. Namely, we can give the following definition.

Let λ be the highest weight of the irreducible representation $\varphi : \mathrm{E}_6 \to \mathfrak{sl}(V)$. Consider the subgroup $H_\lambda < W$ generated by all $w \in W$ such that $\frac{2}{(\alpha_k,\alpha_k)}(w(\lambda+\rho) - (\lambda+\rho), \alpha_k)$ is even, $k = 1,\ldots,\mathrm{rank}(\mathfrak{g})$, that is $\frac{2(w(\lambda+\rho),\alpha_k)}{(\alpha_k,\alpha_k)} \equiv \frac{2(\lambda+\rho,\alpha_k)}{(\alpha_k,\alpha_k)}$ (mod 2). We shall call H_λ the stabilizer of λ. If $w \in H_\lambda$, then we shall write $w(\lambda+\rho) \equiv \lambda+\rho$ (mod 2).

Consider, in accordance with the notations of table 1, the representation $\bar\lambda$ of type $\begin{smallmatrix} \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ & & \mathrm{e} & & \end{smallmatrix}$. That is the type of $\bar\lambda + \rho$ is $\begin{smallmatrix} \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} \\ & & \mathrm{o} & & \end{smallmatrix}$. Consider, furthermore, the co-set decomposition of the Weyl group W with respect to the subgroup $H_{\bar\lambda}$. Then the elements w in lemma 1 represent the co-set classes $W/H_{\bar\lambda}$. For example, consider $\lambda = \sum_{j=1}^{6} \lambda_j \omega_j$ of type $\begin{smallmatrix} \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} \\ & & \mathrm{e} & & \end{smallmatrix}$ and the reflection s_β, where $\beta = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{pmatrix}$. Then $s_\beta(\lambda+\rho) = \lambda + \rho - (\lambda_1+\lambda_2+\lambda_3+2\lambda_4+\lambda_5+\lambda_6+7)\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{pmatrix} = \sum_{j=1}^{6}(\lambda_j+1)\omega_j - (\lambda_1+\lambda_2+\lambda_3+2\lambda_4+\lambda_5+\lambda_6+7)(\omega_1-\omega_3+\omega_4-\omega_5+\omega_6) = \sum_{j=1}^{6} a_j\omega_j$ where the integers $a_j,\ j = 1, 3\text{–}6$ are even and the integer a_2 is odd. Hence, $s_\beta(\lambda+\rho) \equiv \bar\lambda+\rho$ (mod 2). Furthermore, from table 2 it follows that W acts transitive on the set $S_{11} \cup S_{12}$ and it is not necessary to reduce all 36 cases to $\begin{smallmatrix} \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ & & \mathrm{e} & & \end{smallmatrix}$. They are all equivalent. Formula (4) simply would change if we reduce to some other case.

5. The case $\mathfrak{g} = \mathrm{E}_7$, $\mathfrak{g}_\sigma = \mathrm{EVI}$, EVII

Let symbol $\begin{pmatrix} a & c & d & e & f & g \\ & & b & & & \end{pmatrix}$ denote the root $\beta = a\boldsymbol{\alpha}_1 + b\boldsymbol{\alpha}_2 + c\boldsymbol{\alpha}_3 + d\boldsymbol{\alpha}_4 + e\boldsymbol{\alpha}_5 + f\boldsymbol{\alpha}_6 + g\boldsymbol{\alpha}_7$. Let $\mathrm{card}(X(\lambda)) = 28$. If $\mathfrak{g}_\sigma = \mathrm{EVI}$, then from theorem 4 [1] it follows that

$$|\delta| = \frac{C_\lambda}{2^{20} \times 3^9 \times 5^4 \times 7^2} \frac{1}{3!} \left| \sum_{\mu_i} \cos(\pi(\mu_i - \eta_0, \lambda+\rho))(\mu_i, \lambda+\rho)^3 \dim V^{\mu_i} \right| \tag{6}$$

where the sum in (6) embraces all weights μ_i of the representation $\begin{array}{c} 0\quad 0\quad 0\quad 0\quad 0\quad 0 \\ 0-0-0-0-0-0 \\ | \\ 0\ 1 \end{array}$ with the highest weight $\eta_0 = \omega_2$. If $\mathfrak{g}_\sigma = \mathrm{EVII}$, then from theorem 5 [1] it follows that

$$|\delta| = \frac{C_\lambda}{2^{25} \times 3^{10} \times 5^5 \times 7^3 \times 11} \frac{1}{8!} \left| \sum_{\mu_i} \cos(\pi(\mu_i - \eta_0, \lambda + \rho))(\mu_i, \lambda + \rho)^8 \dim V^{\mu_i} \right| \tag{7}$$

where the sum in (7) embraces all weights μ_i of the representation $\begin{array}{c} 1\quad 1\quad 0\quad 0\quad 0\quad 0 \\ 0-0-0-0-0-0 \\ | \\ 0\ 0 \end{array}$ with the highest weight $\eta_0 = \omega_1 + \omega_3$. Transforming formulae (6) and (7)

$$\lambda + \rho = \sum_{j=1}^{7} (\lambda_j + 1)\omega_j = \sum_{j=1}^{8} h_j \varepsilon_j$$

where

$$\begin{aligned}
h_1 &= \tfrac{1}{4}(4\lambda_1 + 7\lambda_2 + 8\lambda_3 + 12\lambda_4 + 9\lambda_5 + 6\lambda_6 + 3\lambda_7 + 49) \\
h_2 &= \tfrac{1}{4}(-\lambda_2 + \lambda_5 + 2\lambda_6 + 3\lambda_7 + 5) \\
h_3 &= \tfrac{1}{4}(-\lambda_2 + \lambda_5 + 2\lambda_6 - \lambda_7 + 1) \\
h_4 &= \tfrac{1}{4}(-\lambda_2 + \lambda_5 - 2\lambda_6 - \lambda_7 - 3) \\
h_5 &= \tfrac{1}{4}(-\lambda_2 - 3\lambda_5 - 2\lambda_6 - \lambda_7 - 7) \\
h_6 &= \tfrac{1}{4}(-\lambda_2 - 4\lambda_4 - 3\lambda_5 - 2\lambda_6 - \lambda_7 - 11) \\
h_7 &= \tfrac{1}{4}(-\lambda_2 - 4\lambda_3 - 4\lambda_4 - 3\lambda_5 - 2\lambda_6 - \lambda_7 - 15) \\
h_8 &= \tfrac{1}{4}(-4\lambda_1 - \lambda_2 - 4\lambda_3 - 4\lambda_4 - 3\lambda_5 - 2\lambda_6 - \lambda_7 - 19).
\end{aligned} \tag{8}$$

Consider the representation $\bar{\lambda}$ of type $\begin{array}{c} \mathrm{o}\quad \mathrm{o}\quad \mathrm{o}\quad \mathrm{o}\quad \mathrm{o}\quad \mathrm{o} \\ \mathrm{e} \end{array}$. That is the type of $\bar{\lambda} + \rho$ is $\begin{array}{c} \mathrm{e}\quad \mathrm{e}\quad \mathrm{e}\quad \mathrm{e}\quad \mathrm{e}\quad \mathrm{e} \\ \mathrm{o} \end{array}$. Discussing this in the same way as before we find $\bar{\lambda} + \rho \simeq (2, 0, 0, 0, 0, 0, 0, 0)$. Furthermore, using the table of all weights μ_i of the representation $\begin{array}{c} 0\quad 0\quad 0\quad 0\quad 0\quad 0 \\ 0-0-0-0-0-0 \\ | \\ 0\ 1 \end{array}$, we derive from (6) the formula

$$|\delta| = \frac{C_{\bar{\lambda}}}{2^{20} \times 3^9 \times 5^4 \times 7^2} \frac{8}{3} \left| \sum_{j=1}^{8} h_j^3 \right|. \tag{9}$$

Similarly, using the table of all weights μ_i of the representation $\begin{array}{c} 1\quad 1\quad 0\quad 0\quad 0\quad 0 \\ 0-0-0-0-0-0 \\ | \\ 0\ 0 \end{array}$, we derive from (7) the formula

$$\begin{aligned}
|\delta| = {} & \frac{C_{\bar{\lambda}}}{2^{25} \times 3^{10} \times 5^5 \times 7^3 \times 11} \\
& \times \left| -128 \sum_{j=1}^{8} h_j^8 + \frac{256}{3} \left(\sum_{j=1}^{8} h_j^6 \right) \left(\sum_{j=1}^{8} h_j^2 \right) + \frac{1088}{15} \left(\sum_{j=1}^{8} h_j^5 \right) \left(\sum_{j=1}^{8} h_j^3 \right) \right.
\end{aligned}$$

$$+16\left(\sum_{j=1}^{8} h_j^4\right)^2 - 24\left(\sum_{j=1}^{8} h_j^4\right)\left(\sum_{j=1}^{8} h_j^2\right)^2$$

$$-\frac{279}{9}\left(\sum_{j=1}^{8} h_j^3\right)^2\left(\sum_{j=1}^{8} h_j^2\right) + \frac{5}{3}\left(\sum_{j=1}^{8} h_j^2\right)^4\Bigg|. \tag{10}$$

Consider the other 35 cases when $\mathrm{card}(X(\lambda')) = 28$ and reduce the case $\mathfrak{g} = \mathrm{E}_7$ to the case $\mathfrak{g} = \mathrm{E}_6$. From table 1 it follows that there exists one-to-one correspondence between the highest weights λ of the algebra $\mathfrak{g} = \mathrm{E}_6$ such that $\mathrm{card}(X(\lambda)) = 16$ and the highest weights λ' of the algebra $\mathfrak{g} = \mathrm{E}_7$ such that $\mathrm{card}(X(\lambda')) = 28$. Namely, any $\lambda = \begin{array}{l}\lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\\ 0-0-0-0-0\\ \qquad\ \ |\\ \qquad\ \ 0\ \lambda_2\end{array} \in S_{11}$ corresponds to $\lambda' = \begin{array}{l}\lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\ \ \mathrm{e}\\ 0-0-0-0-0-0\\ \qquad\ \ |\\ \qquad\ \ 0\ \lambda_2\end{array}$ and any $\lambda = \begin{array}{l}\lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\\ 0-0-0-0-0\\ \qquad\ \ |\\ \qquad\ \ 0\ \lambda_2\end{array} \in S_{12}$ corresponds to $\lambda' = \begin{array}{l}\lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\ \ \mathrm{o}\\ 0-0-0-0-0-0\\ \qquad\ \ |\\ \qquad\ \ 0\ \lambda_2\end{array}$. The foregoing proves the following lemma.

Lemma 2. Consider a regular subalgebra E_6 embedded naturally in E_7. Namely $\begin{array}{l}\alpha_1\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7\\ 0-0-0-0-0-\emptyset\\ \qquad\ \ |\\ \qquad\ \ 0\ \alpha_2\end{array} \subset \begin{array}{l}\alpha_1\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7\\ 0-0-0-0-0-0\\ \qquad\ \ |\\ \qquad\ \ 0\ \alpha_2\end{array}$, where the root α_7 is deleted. Let $\lambda' = \sum_{j=1}^{7}\lambda_j\omega_j = \begin{array}{l}\lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\ \lambda_7\\ 0-0-0-0-0-0\\ \qquad\ \ |\\ \qquad\ \ 0\ \lambda_2\end{array}$ be the highest weight of the representation $\varphi : \mathrm{E}_7 \to \mathfrak{sl}(V)$ and let $\mathrm{card}(X(\lambda')) = 28$. Then for the representation $= \begin{array}{l}\lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\\ 0-0-0-0-0\\ \qquad\ \ |\\ \qquad\ \ 0\ \lambda_2\end{array}$ of the algebra E_6 we have $\mathrm{card}(X(\lambda)) = 16$. Consider the root $\beta = \begin{pmatrix}\beta_1 & \beta_3 & \beta_4 & \beta_5 & \beta_6\\ & & \beta_2 & & \end{pmatrix}$ from table 2 such that $s_\beta(\lambda+\rho) \equiv \begin{matrix}\mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e}\\ & & \mathrm{o} & & \end{matrix}$ (mod 2). Then the root $\beta' = \begin{pmatrix}\beta_1 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & 0\\ & & \beta_2 & & & \end{pmatrix} = \beta + 0\cdot\boldsymbol{\alpha}_7$ defines s_β, such that $s_{\beta'}(\lambda'+\rho) \equiv \begin{matrix}\mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e}\\ & & \mathrm{o} & & & \end{matrix}$ (mod 2). From theorems 4 and 5 [1] and lemma 2 we derive, using formulae (9) and (10), the following theorem.

Theorem 2. Suppose $\lambda = \sum_{j=1}^{7}\lambda_j\omega_j$ is the highest weight of arbitrary representation $\varphi : \mathrm{E}_7 \to \mathfrak{sl}(V)$. Let $\mathfrak{g}_\sigma = \mathrm{EVI}$.

If $\mathrm{card}(X(\lambda)) = 31$, then $|\delta| = \frac{C_\lambda}{2^{16}\times 3^9\times 5^4\times 7^2}$.

If $\mathrm{card}(X(\lambda)) = 36$ or 63, then $\delta = 0$.

If $\mathrm{card}(X(\lambda)) = 28$, then

$$|\delta| = \frac{C_\lambda}{2^{20}\times 3^9\times 5^4\times 7^2}\,\frac{8}{3}\left|\sum_{j=1}^{8} h_j^3\right|. \tag{11}$$

Let $\mathfrak{g}_\sigma = \mathrm{EVII}$.

If $\mathrm{card}(X(\lambda)) = 36$, then $|\delta| = \frac{C_\lambda}{2^{13}\times 3^{10}\times 5^5\times 7^3\times 11}$.

If $\mathrm{card}(X(\lambda)) = 31$ or 63, then $\delta = 0$.

If $\mathrm{card}(X(\lambda)) = 28$, then

$$
\begin{aligned}
|\delta| = {} & \frac{C_\lambda}{2^{25}\times 3^{10}\times 5^5\times 7^3\times 11} \\
& \times\Bigg| -128\sum_{j=1}^{8} h_j^8 + \frac{256}{3}\Bigg(\sum_{j=1}^{8} h_j^6\Bigg)\Bigg(\sum_{j=1}^{8} h_j^2\Bigg) + \frac{1088}{15}\Bigg(\sum_{j=1}^{8} h_j^5\Bigg)\Bigg(\sum_{j=1}^{8} h_j^3\Bigg) \\
& +16\Bigg(\sum_{j=1}^{8} h_j^4\Bigg)^2 - 24\Bigg(\sum_{j=1}^{8} h_j^4\Bigg)\Bigg(\sum_{j=1}^{8} h_j^2\Bigg)^2 \\
& -\frac{279}{9}\Bigg(\sum_{j=1}^{8} h_j^3\Bigg)^2\Bigg(\sum_{j=1}^{8} h_j^2\Bigg) + \frac{5}{3}\Bigg(\sum_{j=1}^{8} h_j^2\Bigg)^4\Bigg| \qquad (12)
\end{aligned}
$$

where h_j, $j = 1, \ldots, 8$ in formulae (11) and (12) are the components of the vector $w(\lambda+\rho)$ in the basis $\varepsilon_1, \ldots, \varepsilon_8$ and $w \in W$ must be defined in accordance with table 2 and lemma 2.

6. The case $\mathfrak{g} = \mathbf{E_8}$, $\mathfrak{g}_\sigma = \mathbf{EIX}$

By symbol $\begin{pmatrix} a & c & d & e & f & g & h \\ & & b & & & & \end{pmatrix}$ denote the root $\beta = a\boldsymbol{\alpha}_1 + b\boldsymbol{\alpha}_2 + c\boldsymbol{\alpha}_3 + d\boldsymbol{\alpha}_4 + e\boldsymbol{\alpha}_5 + f\boldsymbol{\alpha}_6 + g\boldsymbol{\alpha}_7 + h\boldsymbol{\alpha}_8$.

If $\mathrm{card}(X(\lambda)) = 56$, then from [1] it follows that

$$
\begin{aligned}
|\delta| = {} & \frac{C_\lambda}{2^{50}\times 3^{22}\times 5^{10}\times 7^6\times 11^3\times 13^2\times 17} \\
& \times\frac{1}{8!}\Bigg|\sum_{\mu_i}\cos(\pi(\mu_i-\eta_0,\lambda+\rho))(\mu_i,\lambda+\rho)^8\dim V^{\mu_i}\Bigg| \qquad (13)
\end{aligned}
$$

where the sum in (13) embraces all weights μ_i of the representation

```
1   1   0   0   0   0   0
0—0—0—0—0—0—0
        |
        0 0
```

with the highest weight $\eta_0 = \omega_1 + \omega_3$.

Transforming formula (13)

$$\lambda+\rho = \sum_{j=1}^{8}(\lambda_j+1)\omega_j = \sum_{j=1}^{8} h_j\varepsilon_j$$

where

$$
\begin{aligned}
h_1 &= \tfrac{1}{2}(4\lambda_1+5\lambda_2+7\lambda_3+10\lambda_4+8\lambda_5+6\lambda_6+4\lambda_7+2\lambda_8+46) \\
h_2 &= \tfrac{1}{2}(\lambda_2+\lambda_3+2\lambda_4+2\lambda_5+2\lambda_6+2\lambda_7+2\lambda_8+12) \\
h_3 &= \tfrac{1}{2}(\lambda_2+\lambda_3+2\lambda_4+2\lambda_5+2\lambda_6+2\lambda_7+10) \\
h_4 &= \tfrac{1}{2}(\lambda_2+\lambda_3+2\lambda_4+2\lambda_5+2\lambda_6+8) \\
h_5 &= \tfrac{1}{2}(\lambda_2+\lambda_3+2\lambda_4+2\lambda_5+6) \qquad (14) \\
h_6 &= \tfrac{1}{2}(\lambda_2+\lambda_3+2\lambda_4+4) \\
h_7 &= \tfrac{1}{2}(\lambda_2+\lambda_3+2) \\
h_8 &= \tfrac{1}{2}(\lambda_2-\lambda_3).
\end{aligned}
$$

Consider the representation $\bar{\lambda}$ of type $\begin{smallmatrix} \mathrm{e} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ & & \mathrm{o} & & & & \end{smallmatrix}$. That is the type of $\bar{\lambda}+\rho$ is $\begin{smallmatrix} \mathrm{o} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} \\ & & \mathrm{e} & & & & \end{smallmatrix}$. Discussing this in the same way as previously we find $\bar{\lambda}+\rho \simeq (2,0,0,0,0,0,0,0)$. Then from (13) we derive

$$|\delta| = \frac{C_{\bar{\lambda}}}{2^{50}\times 3^{22}\times 5^{10}\times 7^{6}\times 11^{3}\times 13^{2}\times 17}\,\frac{2}{3} \times\left| -48\sum_{j=1}^{8} h_j^8 + 32\left(\sum_{j=1}^{8} h_j^6\right)\left(\sum_{j=1}^{8} h_j^2\right) + 108\left(\sum_{j=1}^{8} h_j^4\right)^2 -60\left(\sum_{j=1}^{8} h_j^4\right)\left(\sum_{j=1}^{8} h_j^2\right)^2 + 7\left(\sum_{j=1}^{8} h_j^2\right)^4 - 6528 h_1h_2h_3h_4h_5h_6h_7h_8\right|. \tag{15}$$

Let $\lambda' = \begin{array}{c} \lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\ \lambda_7\ \lambda_8 \\ 0\text{—}0\text{—}0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array}$. From table 1 it follows that there are 135 cases when $\mathrm{card}(X(\lambda')) = 56$.

6.1.

Consider 108 cases when $\begin{array}{c} \lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6 \\ 0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array} \in S_{11}$ or S_{12}.

Lemma 3. Let the type of $\bar{\lambda}$ be $\begin{smallmatrix} \mathrm{e} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} \\ & & \mathrm{o} & & & & \end{smallmatrix}$.

If $\gamma = \begin{pmatrix} 1 & 3 & 5 & 4 & 3 & 2 & 1 \\ & & 3 & & & & \end{pmatrix}$, then $s_\gamma(\bar{\lambda}+\rho) \equiv \begin{smallmatrix} \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} \\ & & \mathrm{o} & & & & \end{smallmatrix} \pmod 2$.

If $\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & & & & \end{pmatrix}$, then $s_\gamma(\bar{\lambda}+\rho) \equiv \begin{smallmatrix} \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{o} \\ & & \mathrm{o} & & & & \end{smallmatrix} \pmod 2$.

If $\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ & & 0 & & & & \end{pmatrix}$, then $s_\gamma(\bar{\lambda}+\rho) \equiv \begin{smallmatrix} \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{e} & \mathrm{o} & \mathrm{o} \\ & & \mathrm{o} & & & & \end{smallmatrix} \pmod 2$.

Lemma 4. Consider a regular subalgebra E_6 embedded naturally in E_8. Namely $\begin{array}{c} \alpha_1\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7\ \alpha_8 \\ 0\text{—}0\text{—}0\text{—}0\text{—}0\text{—}\emptyset\text{—}\emptyset \\ | \\ 0\ \alpha_2 \end{array} \subset \begin{array}{c} \alpha_1\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7\ \alpha_8 \\ 0\text{—}0\text{—}0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \alpha_2 \end{array}$, where the roots α_7 and α_8 are deleted. Let $\lambda' = \begin{array}{c} \lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6\ \lambda_7\ \lambda_8 \\ 0\text{—}0\text{—}0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array}$ be the highest weight of the representation $\varphi : E_8 \to \mathfrak{sl}(V)$ such that $\mathrm{card}(X(\lambda')) = 56$. Ignore the marks λ_7 and λ_8 and suppose $\lambda = \begin{array}{c} \lambda_1\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6 \\ 0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array} \in S_{11}$ or S_{12}.

Let $w \in W$ correspond to λ in table 2. If $\lambda \in S_{11}$ and the marks λ_7 and λ_8 are even or $\lambda \in S_{12}$ and λ_7 is odd and λ_8 is even, then $s_\gamma(w(\lambda'+\rho)) \equiv \bar{\lambda}+\rho$ (mod

2), where $\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & & & & \end{pmatrix}$. If $\lambda \in S_{11}$ and λ_7 is even and λ_8 is odd or $\lambda \in S_{12}$ and the marks λ_7 and λ_8 are odd, then $s_\gamma(w(\lambda' + \rho)) \equiv \bar{\lambda} + \rho$ (mod 2), where $\gamma = \begin{pmatrix} 1 & 3 & 5 & 4 & 3 & 2 & 1 \\ & & 3 & & & & \end{pmatrix}$. If $\lambda \in S_{11}$ and λ_7 is odd and λ_8 is even or $\lambda \in S_{12}$ and the marks λ_7 and λ_8 are even, then $s_\gamma(w(\lambda' + \rho)) \equiv \bar{\lambda} + \rho$ (mod 2), where $\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ & & 0 & & & & \end{pmatrix}$.

Lemmas 3 and 4 are derived by straightforward calculation.

6.2.

Consider 16 cases when $\lambda^* = \begin{array}{c} \lambda_1 \;\; \lambda_3 \;\; \lambda_4 \;\; \lambda_5 \;\; \lambda_6^* \\ 0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array} \in S_{22}$.

From table 1 it follows that if $\lambda^* = \begin{array}{c} \lambda_1 \;\; \lambda_3 \;\; \lambda_4 \;\; \lambda_5 \;\; \lambda_6^* \\ 0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array} \in S_{22}$ and if we change the evenness of λ_6^*, then we obtain $\lambda = \begin{array}{c} \lambda_1 \;\; \lambda_3 \;\; \lambda_4 \;\; \lambda_5 \;\; \lambda_6 \\ 0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array} \in S_{12}$ and *vice versa*. Discussing this in the same way as before we derive the following lemma.

Lemma 5. Let the type of $\bar{\lambda}$ be $\begin{array}{ccccccc} e & o & o & o & o & o & o \\ & & o & & & & \end{array}$.

If $\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ & & 0 & & & & \end{pmatrix}$, then $s_\gamma(\bar{\lambda} + \rho) \equiv \begin{array}{ccccccc} e & e & e & e & o & o & e \\ & & o & & & & \end{array}$ (mod 2).

Let $\lambda^* = \begin{array}{c} \lambda_1 \;\; \lambda_3 \;\; \lambda_4 \;\; \lambda_5 \;\; \lambda_6^* \\ 0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array} \in S_{22}$ and let $\lambda = \begin{array}{c} \lambda_1 \;\; \lambda_3 \;\; \lambda_4 \;\; \lambda_5 \;\; \lambda_6 \\ 0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array} \in S_{12}$ be the highest weight with the changed evenness of λ_6^*. Let $w \in W$ correspond to λ in table 2.

If $\lambda' = \begin{array}{c} \lambda_1 \;\; \lambda_3 \;\; \lambda_4 \;\; \lambda_5 \;\; \lambda_6^* \;\; e \;\; o \\ 0\text{—}0\text{—}0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array}$, then $s_\gamma(w(\lambda' + \rho)) \equiv \bar{\lambda} + \rho$ (mod 2).

Hence in the cases S_{11}, S_{12} and S_{22} we derive, using lemmas 3–5, table 3 for the definition of $w \in W$ such that $w(\lambda' + \rho) \equiv \bar{\lambda} + \rho$ (mod 2).

6.3.

Consider 11 cases when $\begin{array}{c} \lambda_1 \;\; \lambda_3 \;\; \lambda_4 \;\; \lambda_5 \;\; \lambda_6 \\ 0\text{—}0\text{—}0\text{—}0\text{—}0 \\ | \\ 0\ \lambda_2 \end{array} \in S_{21}$.

Discussing this in the same way as we have done previously we derive table 4 for the definition of $w \in W$ such that $w(\lambda' + \rho) \equiv \bar{\lambda} + \rho$ (mod 2).

From theorem 7 [1] and the foregoing results we derive the following theorem.

Table 3. The elements $w \in W$ in the cases $\lambda \in S_{11}$, S_{12}, S_{22}. The integers λ_6 and λ_6^* in the case S_{22} satisfy the congruence $\lambda_6 \equiv \lambda_6^* + 1 \pmod 2$. If $\mathrm{card}(X(\lambda)) = 20$ for $\mathrm{g} = \mathrm{E}_6$ or $\mathrm{card}(X(\lambda)) = 31$ for $\mathrm{g} = \mathrm{E}_7$, then $w \in W$ is not necessary for these algebras.

λ_1	λ_3	λ_4 λ_2	λ_5	λ_6	$\mathsf{g} = \mathrm{E}_6$ $w \in W$	λ_7	$\mathsf{g} = \mathrm{E}_7$ $w \in W$	λ_8	$\mathsf{g} = \mathrm{E}_8$ $w \in W$
S_{11}					w from table 2	e	w from table 2	e	$\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & & & & \end{pmatrix}$, w from table 2 $s_\gamma \circ w$
								o	$\gamma = \begin{pmatrix} 1 & 3 & 5 & 4 & 3 & 2 & 1 \\ & & 3 & & & & \end{pmatrix}$, w from table 2 $s_\gamma \circ w$
						o	card=31	e	$\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ & & 0 & & & & \end{pmatrix}$, w from table 2 $s_\gamma \circ w$
S_{12}					w from table 2	o	w from table 2	e	$\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & & & & \end{pmatrix}$, w from table 2 $s_\gamma \circ w$
								o	$\gamma = \begin{pmatrix} 1 & 3 & 5 & 4 & 3 & 2 & 1 \\ & & 3 & & & & \end{pmatrix}$, w from table 2 $s_\gamma \circ w$
						e	card=31	e	$\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ & & 0 & & & & \end{pmatrix}$, w from table 2 $s_\gamma \circ w$
S_{22}					card=20	e	card=31	o	$\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ & & 0 & & & & \end{pmatrix}$ $s_\gamma \circ w$ w from table 2 for λ_1 λ_3 λ_4 λ_2 λ_5 λ_6^*

Theorem 3. Suppose $\lambda = \sum_{j=1}^{8} \lambda_j \omega_j$ is the highest weight of arbitrary representation $\varphi : \mathrm{E}_8 \to \mathfrak{sl}(V)$. let $\mathfrak{g}_\sigma = \mathrm{EIX}$.

If $\mathrm{card}(X(\lambda)) = 64$, then $|\delta| = \frac{C_\lambda}{2^{36} \times 3^{22} \times 5^{10} \times 7^6 \times 11^3 \times 13^2 \times 17}$.

If $\mathrm{card}(X(\lambda)) = 120$, then $\delta = 0$.

If $\mathrm{card}(X(\lambda)) = 56$, then

$$|\delta| = \frac{C_\lambda}{2^{50} \times 3^{22} \times 5^{10} \times 7^6 \times 11^3 \times 13^2 \times 17} \frac{2}{3} \times \left| -48 \cdot \sum_{j=1}^{8} h_j^8 + 32\left(\sum_{j=1}^{8} h_j^6\right)\left(\sum_{j=1}^{8} h_j^2\right) + 108\left(\sum_{j=1}^{8} h_j^4\right)^2 - 60\left(\sum_{j=1}^{8} h_j^4\right)\left(\sum_{j=1}^{8} h_j^2\right)^2 + 7\left(\sum_{j=1}^{8} h_j^2\right)^4 - 6528 h_1 h_2 h_3 h_4 h_5 h_6 h_7 h_8 \right| \tag{16}$$

Table 4. The elements $w \in W$ in the case $\lambda \in S_{21}$. Symbols e(o,a) have the same meaning as in tables 1–4.

The type of λ	o o o e o o o e	e e o o e o o o	o o e e e o o e	o e e o e o o o
The root β	$\begin{pmatrix}0&1&2&2&1&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}0&0&1&1&1&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}0&1&2&1&1&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}0&1&1&1&1&0&0\\&&1&&&&\end{pmatrix}$
The root γ	$\begin{pmatrix}1&2&3&2&1&0&0\\&&2&&&&\end{pmatrix}$	$\begin{pmatrix}1&2&3&2&1&0&0\\&&2&&&&\end{pmatrix}$	$\begin{pmatrix}1&2&3&2&1&0&0\\&&2&&&&\end{pmatrix}$	$\begin{pmatrix}1&2&3&2&1&0&0\\&&2&&&&\end{pmatrix}$
$w \in W$	$s_\gamma \circ s_\beta$	$s_\gamma \circ s_\beta$	$s_\gamma \circ s_\beta$	$s_\gamma \circ s_\beta$
The type of λ	o o o e e o o e	o e e o o o o o	o o e e o o o e	e e o o o o o o
The root β_1	$\begin{pmatrix}0&1&2&1&1&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}1&2&2&2&1&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}0&1&2&2&1&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}1&1&2&1&0&0&0\\&&1&&&&\end{pmatrix}$
The root β_2	$\begin{pmatrix}0&0&0&0&0&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}1&1&1&0&0&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}1&1&1&1&1&0&0\\&&1&&&&\end{pmatrix}$
The root γ	$\begin{pmatrix}1&2&3&2&1&0&0\\&&2&&&&\end{pmatrix}$	$\begin{pmatrix}1&2&3&2&1&0&0\\&&2&&&&\end{pmatrix}$	$\begin{pmatrix}1&2&3&2&1&0&0\\&&2&&&&\end{pmatrix}$	$\begin{pmatrix}1&2&3&2&1&0&0\\&&2&&&&\end{pmatrix}$
$w \in W$	$s_\gamma \circ s_{\beta_2} \circ s_{\beta_1}$	$s_\gamma \circ s_{\beta_2} \circ s_{\beta_1}$	$s_\gamma \circ s_{\beta_2} \circ s_{\beta_1}$	$s_\gamma \circ s_{\beta_2} \circ s_{\beta_1}$
The type of λ	e o o o e o o o	o o o o e o o o	e o o o o o o o	
The root β	$\begin{pmatrix}1&1&1&1&0&0&0\\&&0&&&&\end{pmatrix}$	$\begin{pmatrix}1&1&1&1&1&0&0\\&&1&&&&\end{pmatrix}$	—	
The root γ	$\begin{pmatrix}1&1&1&0&0&0&0\\&&1&&&&\end{pmatrix}$	$\begin{pmatrix}1&2&3&2&1&0&0\\&&1&&&&\end{pmatrix}$	—	
$w \in W$	$s_\gamma \circ s_\beta$	$s_\gamma \circ s_\beta$	Id	

where $h_j,\ j = 1, \ldots, 8$ in formula (16) are the components of the vector $w(\lambda + \rho)$ in the basis $\varepsilon_1, \ldots, \varepsilon_8$ and $w \in W$ must be defined in accordance with tables 3 and 4.

7. Examples

Consider the representation $\lambda = \begin{array}{c} 0\quad 1\quad 0\quad 0\quad 1 \\ 0—0—0—0—0 \\ | \\ 0\ 0 \end{array}$ of the algebra E_6.

Let $\mathfrak{g}_\sigma = \mathrm{EIII}$. Then $\mathrm{card}(X(\lambda)) = 16$, $\lambda \in S_{11}$ and $C_\lambda = 1\,451\,520$. From table 2 it follows that $w(\lambda + \rho) = s_{\beta_2} \circ s_{\beta_1}(\lambda + \rho)$, where $\beta_1 = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 \\ & & 1 & & \end{pmatrix}$ and $\beta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & & 1 & & \end{pmatrix}$. Hence from (2) we derive $w(\lambda + \rho) = \frac{1}{3}(6, 25, -5, -11, -17, 7, 1, -6)$ and from (5) it follows that $|\delta| = 75$. Consider the representation $\lambda = \begin{array}{c} 0\quad 1\quad 0\quad 0\quad 1\quad 0 \\ 0—0—0—0—0—0 \\ | \\ 0\ 0 \end{array}$ of the algebra E_7. Then $\mathrm{card}(X(\lambda)) = 28$ and $C_\lambda = 2^{19} \times 3^9 \times 5^3 \times 7^2 \times 11$. From tables 3 and 2 it follows that $w(\lambda+\rho) = s_{\beta_2} \circ s_{\beta_1}(\lambda+\rho)$, where $\beta_1 = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 0 \\ & & 1 & & & \end{pmatrix}$ and $\beta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & & & \end{pmatrix}$. Hence from (8) we derive $w(\lambda + \rho) = \frac{1}{4}(43, 27, -13, -21, 11, 3, -5, -45)$. Let $\mathfrak{g}_\sigma = \mathrm{EVI}$, then from (11) it follows that $|\delta| = 99$. Let $\mathfrak{g}_\sigma = \mathrm{EVII}$, then from (12) it follows that $|\delta| = 7371$.

Consider the representation $\lambda = \begin{array}{c} 0\quad 1\quad 0\quad 0\quad 1\quad 0\quad 0 \\ 0—0—0—0—0—0—0 \\ | \\ 0\ 0 \end{array}$ of the algebra E_8. Let $\mathfrak{g}_\sigma = \mathrm{EIX}$. Then $\mathrm{card}(X(\lambda)) = 56$ and $C_\lambda = 2^{42} \times 3^{22} \times 5^8 \times 7^7 \times 11^3 \times 13^2$. From tables 3 and 2 it follows that $w(\lambda+\rho) = s_\gamma \circ s_{\beta_2} \circ s_{\beta_1}(\lambda+\rho)$, where $\beta_1 = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ & & 1 & & & & \end{pmatrix}$, $\beta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & & & & \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & & & & \end{pmatrix}$. Hence from (14) we derive $w(\lambda + \rho) = (20, -2, 16, 6, 4, 12, 10, 8)$ and from (16) it follows that $|\delta| = 2\,046\,870$.

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